Category-Theoretic Reconstruction of Schemes from Categories of Reduced Schemes

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Let $\mathbf{U}$ and $\mathbf{V}$ be Grothendieck universes such that $\mathbf{U} \in \mathbf{V}$. Let S be a $\mathbf{U}$-small scheme. In the following, we shall use the term "scheme" to refer to a $\mathbf{U}$-small scheme. Let $\blacklozenge/S$ be a ($\mathbf{V}$-small) set of properties of S-schemes. We shall write

$$\mathsf{Sch}_{\blacklozenge/S}$$

for the full subcategory of the ($\mathbf{V}$-small) category of S-schemes $\mathsf{Sch}_{/S}$ determined by the objects $X \in \mathsf{Sch}_{\blacklozenge/S}$ that satisfy every property of $\blacklozenge/S$. In [YJ], we shall mainly be concerned with the properties

"red", "qcpt", "qsep", "sep"

of S-schemes, i.e., "reduced", "quasi-compact over S", "quasi-separated over S", and "separated over S". If $\blacklozenge/S = \varnothing$, then we simply write $\mathsf{Sch}_{/S}$ for $\mathsf{Sch}_{\blacklozenge/S}$. In [YJ], **we consider the problem of reconstructing the scheme S from the intrinsic structure of the abstract category $\mathsf{Sch}_{\blacklozenge/S}$.** In [Mzk04], Mochizuki gave a solution to this problem in the case where S is locally Noetherian, and $\blacklozenge/S$ = "locally of finite type over S". In [vDdB19], van Dobben de Bruyn gave a solution to this problem in the case where S is arbitrary scheme, and $\blacklozenge/S = \varnothing$. The techniques applied in [vDdB19] make essential use of the existence of *non-reduced schemes* in $\mathsf{Sch}_{/S}$. By contrast, in [YJ], we focus on the problem of reconstructing the scheme S from categories of S-schemes that only contain *reduced S-schemes*, hence rely on techniques that differ essentially from the techniques applied in [vDdB19].

If X, Y are objects of a ($\mathbf{V}$-small) category $\mathcal{C}$, then we shall write $\mathrm{Isom}(X, Y)$ for the set of isomorphisms from X to Y. By a slight abuse of notation, we shall also regard this set as a discrete category. If $\mathcal{C}, \mathcal{D}$ are ($\mathbf{V}$-small) categories, then we shall write $\mathbf{Isom}(\mathcal{C}, \mathcal{D})$ for the ($\mathbf{V}$-small) category of equivalences $\mathcal{C} \xrightarrow{\sim} \mathcal{D}$ and natural isomorphisms. If $\mathcal{C}$ is a ($\mathbf{V}$-small) category, and X is an object of $\mathcal{C}$, then we shall write $\mathcal{C}_{/X}$ for the slice category of objects and morphisms equipped with a structure morphism to X. If $f : X \to Y$ is a morphism in a ($\mathbf{V}$-small) category $\mathcal{C}$ which is closed under fiber products, then we shall write $f^* : \mathcal{C}_{/Y} \to \mathcal{C}_{/X}$ for the functor induced by the operation of base-change, via f, from X to Y. The main result in [YJ] is the following:

Main Theorem.

1. *Let S be a normal locally Noetherian ($\mathbf{U}$-small) schemes, $\blacklozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$ a [possibly empty] subset. Then the following may be constructed category-theoretically from $\mathsf{Sch}_{\blacklozenge/S}$ by means of algorithms that are independent of the choice of the subset $\blacklozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$:*
   (a) *for each object T of $\mathsf{Sch}_{\blacklozenge/S}$, a ($\mathbf{V}$-small) scheme $T_{\mathbf{V}}$ and an isomorphism of ($\mathbf{V}$-small) schemes $\varphi_T : T \to T_{\mathbf{V}}$ (where we note that a $\mathbf{U}$-small scheme is, in particular, $\mathbf{V}$-small), and*
   (b) *for each morphism $f : T_1 \to T_2$ of $\mathsf{Sch}_{\blacklozenge/S}$, a morphism of ($\mathbf{V}$-small) schemes $f_{\mathbf{V}} : T_{1,\mathbf{V}} \to T_{2,\mathbf{V}}$ such that $\varphi_{T_2} \circ f = f_{\mathbf{V}} \circ \varphi_{T_1}$.*
2. *Let S, T be normal locally Noetherian ($\mathbf{U}$-small) schemes, $\blacklozenge, \lozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$ [possibly empty] subsets such that $\{\mathrm{qsep}, \mathrm{sep}\} \not\subset \blacklozenge$, $\{\mathrm{qsep}, \mathrm{sep}\} \not\subset \lozenge$. If the ($\mathbf{V}$-small) categories $\mathsf{Sch}_{\blacklozenge/S}$, $\mathsf{Sch}_{\lozenge/T}$ are equivalent, then $\blacklozenge = \lozenge$.*
3. *Let S, T be ($\mathbf{U}$-small) disjoint unions of quasi-separated normal integral ($\mathbf{U}$-small) schemes, $\blacklozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$ a [possibly empty] subset. Then the natural functor*
$$\begin{aligned} \mathrm{Isom}(S, T) &\to \mathbf{Isom}(\mathsf{Sch}_{\blacklozenge/T}, \mathsf{Sch}_{\blacklozenge/S}) \\ f &\mapsto f^* \end{aligned}$$
is an equivalence of ($\mathbf{V}$-small) categories.

References

[Mzk04] S. Mochizuki, *Categorical representation of locally Noetherian log schemes*. Adv. Math. **188** (2004), no.1, 222–246.

[vDdB19] R. van Dobben de Bruyn, *Automorphisms of Categories of Schemes*. Preprint, arXiv:1906.00921.

[YJ] T. Yuji, *Category-Theoretic Reconstruction of Schemes from Categories of Reduced Schemes*. Preprint.